

Anderson transitions in disordered two-dimensional lattices

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We numerically analyze the energy level statistics of the Anderson model with Gaussian site disorder and constant hopping. The model is realized on different two-dimensional lattices, namely, the honeycomb, the kagomé, the square, and the triangular lattice. By calculating the well-known statistical measures viz., nearest neighbor spacing distribution, number variance, the partition number and the dc electrical conductivity from Kubo-Greenwood formula, we show that there is clearly a delocalization to localization transition with increasing disorder. Though the statistics in different lattice systems differs when compared with respect to the change in the disorder strength only, we find there exists a single complexity parameter, a function of the disorder strength, coordination number, localization length, and the local mean level spacing, in terms of which the statistics of the fluctuations matches for all lattice systems at least when the Fermi energy is selected from the bulk of the energy levels.

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I. INTRODUCTION

The effect of impurities in the low dimensional systems such as Graphene^{1–4}, nano flakes⁵, metallic thin films^{6,7}, arrays of quantum dots etc., has been an intense area of research in the last decade. Some of these low dimensional structures have potential applications in electronic devices, because of their finite size and the performance under varying degree of disorder and dimensionality^{7,8}. In disordered systems delocalization to localization transition is one of the interesting phenomenon, therefore to characterize this transition, analysis of the electronic wavefunction is necessary. The extended to localized transition, also known as the ‘Anderson transition’, and its dependence on various system parameters such as, disorder, dimensionality, lattice structure, system size etc., in the finite systems are a frontier area of research. This motivates us to seek an estimation of the critical disorder for the transition in finite size lattices with different geometry.

In 1958, Anderson suggested that an electron inside a material can be fully localized in the presence of a large disorder⁹, whereas Edwards *et al.* showed that a transition from an extended to a localized states in the square lattice can occur at a disorder strength of 5 or 6 times higher than the band width of the Anderson model Hamiltonian¹⁰. In 1979, Abrahams *et al.* conjectured, based on a scaling hypothesis, that the electronic states are localized in less than three dimensional (3D) systems in the presence of any amount of disorder^{11,12}. The scaling hypothesis was later on supported by many studies of the Anderson model on square lattices¹³. Further, Altshuler *et al.* showed that weak electron interaction enhances the localization in these systems, whereas under strong electron interaction, 2D electrons behave like a Wigner crystal and, as shown by Tanatar *et al.*¹⁴, even a small amount of disorder can make the system insulat-

ing at zero temperature. The theoretical analysis¹⁵ suggested that under strong electron interactions and small disordered regime, 2D systems can be conducting. Most of these studies are done for infinite systems and square lattices, except some of the recent studies on the honeycomb lattices^{4,16}. A detailed review on the Anderson transition is given in the Ref. 17.

The study of Anderson transition is not only important for material sciences but also relevant to understand the influence of the wavefunction dynamics on the physical properties of the disordered systems. The presence of disorder and/or interaction leads to a randomization of the Hamiltonian, resulting in a random matrix representation in a physically suitable basis e.g., site basis. The structure of the matrix e.g., degree of sparsity is sensitive to various system conditions viz., dimensionality, shape, size, and boundary conditions. The statistical behaviour of the system can therefore be analysed by an ensemble of the disordered Hamiltonians. Such analysis has been a subject of extensive study during the past decade. It is now well-known that, in the weak disorder regime, the statistics can be well-modelled by the Wigner-Dyson universality classes of random matrix ensembles which correspond to extended, featureless eigenfunctions and a strong level-repulsion with statistics of the eigenvalues and the eigenfunction independent from each other. Increasing the disorder in finite size systems causes the statistics to crossover from Wigner-Dyson to the Poisson universality class (with no level-repulsion and fully localized eigenfunctions in strong disorder limit). The statistics in the intermediate regime e.g., near critical disorder is sensitive to the degree of eigenfunction localization which in turn is expected to depend on the system conditions besides disorder. The study^{17,18} however, showed that the statistics can be well-modeled by the single parametric power-law random banded matrix (PRBM) ensembles. Another theoretical study¹⁹ later

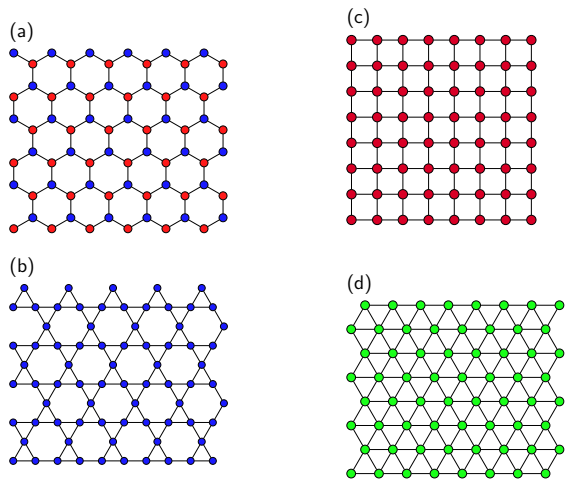


FIG. 1. (Color online) Two dimensional lattice structures considered in this paper: (a) the honeycomb lattice with coordination number 3, (b) the kagomé and (c) the square lattices both having 4 nearest neighbors and (d) the triangular lattice with coordination number 6

on indicated the application of a wide range of random matrix ensembles (besides PRBM) as the model for the intermediate statistics; this study was based on the common mathematical formulation of the energy level statistics of a broad class of random matrix ensembles (with varying degree of sparsity and disorder but same symmetry class). The formulation is governed by a single parameter, referred as the complexity parameter, a function of all system parameters including energy range of interest and therefore different ensembles are expected to show analogous statistics if their complexity parameters are same¹⁹. As the theoretical claim about the existence of a complexity parameter is in clear agreement with the single parameter scaling conjecture of Ref. 11, it is highly desirable to seek its numerical validity in disordered systems.

In this paper, we study the transitions from extended to localized state of Anderson Hamiltonian on finite 2D systems with different geometry in presence of the disorder. As shown in Fig. 1, we consider four lattice geometries, namely, square, triangular, honeycomb and kagomé lattice which differ from each other in the coordination number and their bond connectivity. The coordination numbers of honeycomb, square, kagomé and triangular, lattice are three, four, four, and six respectively. We note that the square and the kagomé both have four nearest neighbour but their spectrum is completely different because of the bond connectivity. The paper is organized as follows. In section II, we introduce the model disordered Hamiltonian; here the theory leading to single parameter based formulation of the spectral statistics is also briefly reviewed. The section III briefly describes the statistical measures used in our numerical analysis. The section IV presents the details of the numerical techniques as well as results; this section is divided into four

sub-sections one for each lattice system. The comparison of the statistics of physical parameters with respect to the single complexity parameter for various lattices is given in section V. In section VI, results are discussed and compared with the existing literature.

II. THE MODEL

The Hamiltonian: We consider the standard Anderson model⁹ under tight binding approximation realized on four different 2D lattice systems. The tight binding model Hamiltonian is well suited for a metallic system, where electron-electron interactions are screened. The Hamiltonian can be written in a single particle basis as

$$H = \sum_i \epsilon_i |i\rangle\langle i| + \sum_{\langle i,j \rangle} t_{ij} |i\rangle\langle j|, \quad (1)$$

where ϵ_i , and t_{ij} are random site energies and the nearest neighbour hopping energies. In this work, the sites energies are considered Gaussian distributed for each lattice system and the hopping energies are kept constant: $t_{ij} = -1.0$. The Gaussian type site disorder has been widely studied for the square lattice system and has a close analogy with real thin film materials.

The ensemble: The presence of the disorder in the system makes it inevitable to consider an ensemble of the Hamiltonians. For the Hamiltonian in Eq.(1) with on-site random energies, the probability density $\rho(H)$ of the ensemble including all possibilities can be written in general as

$$\rho(H) = C \exp \left[- \sum_k \frac{1}{2h_{kk}} (H_{kk} - b_{kk})^2 \right] \times \prod_{k,l; k \neq l} \delta(H_{kl} - f_{kl}), \quad (2)$$

where C is the normalization constant, $f_{kl} = t$ if $\{k, l\}$ correspond to nearest neighbour sites and otherwise $f_{kl} = 0$. Writing the delta function as a limiting Gaussian ($\delta(x) = \lim_{v \rightarrow 0} \frac{1}{\sqrt{2\pi v^2}} e^{-x^2/2v^2}$), reduces Eq.(2) to the following form

$$\rho(H, h, b) = C \exp \left[\sum_{k \leq l} \frac{1}{h_{kl}} (H_{kl} - b_{kl})^2 \right], \quad (3)$$

where h is the set of all variances $h_{kl} = \langle H_{kl}^2 \rangle - \langle H_{kl} \rangle^2$, and b is the set of all mean values $b_{kl} = \langle H_{kl} \rangle$. Here, for constant hopping between nearest neighbor sites and an onsite Gaussian disorder W in Eq.(1), one has

$$h_{kk} = W^2/12, \quad b_{kk} = 0, \quad (4)$$

$$h_{kl} = 0, \quad b_{kl} = -f(kl)t, \quad (5)$$

where $f(kl) = 1$ for $\{k, l\}$ pairs representing nearest neighbors and zero otherwise. The statistical behavior of

the Hamiltonian H , Eq.(1), can now be analyzed using the ensemble (3). An important aspect of Eq.(3) is that it can represent the ensemble density for a wider class of lattice systems under different conditions e.g., random, anisotropic, variable range hopping, dimensionality and boundary conditions; the only constraint on these system is to preserve the time-reversal symmetry which allows H to be real-symmetric. Its form is therefore appropriate for the verification of single parameter scaling conjecture.

Single parameter formulation: Before proceeding to numerical analysis, it is helpful to briefly review the complexity parametric formulation of the spectral statistics. The variation of system conditions in general result in a variation of the distribution parameters h and b and therefore an evolution of the H -ensemble. As discussed in Ref. 19 and 20, under a change of parameters $h_{kl} \rightarrow h_{kl} + \delta h_{kl}$ and $b_{kl} \rightarrow b_{kl} + \delta b_{kl}$, $\rho(H)$ undergoes a diffusion dynamics along with a finite drift, and, using Gaussian nature of $\rho(H)$, it can exactly be shown that

$$T\rho = L\rho, \quad (6)$$

where T is a combination of parametric derivatives, and L is a diffusion operator in matrix space and are given by

$$T = \sum_{k \leq l} \left[(g_{kl} - 2h_{kl}) \frac{\partial}{\partial h_{kl}} - b_{kl} \frac{\partial}{\partial b_{kl}} \right] \quad (7)$$

and

$$L = \sum_{kl} \frac{\partial}{\partial H_{kl}} \left[\frac{g_{kl}}{2} \frac{\partial}{\partial H_{kl}} + H_{kl} \right] \quad (8)$$

with $g_{kl} = 1 + \delta_{kl}$. A suitable transformation of parametric space maps T to a single parametric derivative, $T\rho = \frac{\partial \rho}{\partial Y}$, which in turn reduces Eq. (6) to a single parametric diffusion equation^{19,20}

$$\frac{\partial \rho}{\partial Y} = L\rho. \quad (9)$$

Here

$$Y = \frac{-1}{N^2} \sum_{k \leq l} \left[\ln |1 - (2 - \delta_{kl})h_{kl}| + \ln |b_{kl} + \delta_{b0}|^2 \right] + C_y \quad (10)$$

with C_y as an arbitrary constant and $\delta_{b0} = 1$ if $b_{kl} = 0$ else $\delta_{b0} = 0$. For the ensemble described by the set of parameters in Eq.(5), this leads to

$$Y = \frac{-1}{N} \ln \left[\left| 1 - \frac{W^2}{12} \right| \cdot |t + \delta_{t0}|^{z/2} \right] + C_y, \quad (11)$$

where z is the coordination number of the lattices.

The joint probability distribution of the eigenvalues $P(\{E_n\}) \equiv P(E_1, \dots, E_N)$ for a metal (fully extended eigenfunctions limit) is given by the Wigner-Dyson distribution, $P(\{E_n\}) = \prod_{i < j} |E_i - E_j| \exp(-\frac{1}{2} \sum_k E_k^2)$. The

eigenvalues are uncorrelated in insulator limit, thus implying $P(\{E_n\}) = \prod_{i < j} P_n(E_n)^{21}$. The distribution for the intermediate states of localization can be derived by integrating ρ over the associated eigenvector space. As shown in Ref. 19, an integration of Eq.(9) leads to a single parametric diffusion of the eigenvalue distribution of the ensemble (3)

$$\frac{\partial P}{\partial Y} = \sum_n \frac{\partial}{\partial E_n} \left[\frac{\partial}{\partial E_n} + \sum_{m \neq n} \frac{1}{E_m - E_n} + E_n \right] P. \quad (12)$$

The above equation can be used to obtain the ensemble averaged level density as well as its local fluctuations¹⁹. As discussed in Ref. 22, while the diffusion of the average level density is governed by Y , the diffusion of its fluctuations occurs at a scale determined by $(Y - Y_0) \sim \Delta_l^2$ where Y_0 is the value of Y at the beginning of evolution and Δ_l is the local mean level spacing:

$$\Delta_{local}(E) \approx L^d \xi^{-d} \Delta(E) \quad (13)$$

with ξ as the localization length and d as the dimension (here $d = 2$). The statistics other than mean level density is therefore governed by a rescaled parameter $\Lambda(E)$:

$$\Lambda(E) = \frac{|Y - Y_0|}{\Delta_l^2}. \quad (14)$$

The solution of Eq.(12) at $\Lambda \rightarrow \infty$ corresponds to the Wigner-Dyson statistics. The $\Lambda \rightarrow 0$ limit correspond to the distribution at the initial state of the evolution; for an insulator initial state, the spectral statistics reduces to the statistics of uncorrelated energy levels.

The transition parameter Λ is in general a function of various parameters e.g., disorder, system size, dimensionality, energy range of interest, lattice topology. Although both Y and Δ_l contribute to the system dependence of Λ , the crucial influence comes from Δ_l due to its dependence on the localization length ξ . For finite system sizes N , a variation of system conditions e.g., disorder leads to a smooth crossover of statistics between the stationary limits $\Lambda \rightarrow 0$ and $\Lambda \rightarrow \infty$. In infinite size limit, the statistics abruptly changes from one stationary limit to the other. If however, the limit $\Lambda^* = \lim_{N \rightarrow \infty} \Lambda(N)$ exists, the corresponding statistics would belong to a universality class different from the two stationary limits. The existence of Λ^* therefore is a criteria for the existence of critical spectral statistics¹⁹.

III. FLUCTUATION MEASURES: THE DEFINITIONS

Our main objectives in this paper is to study the influence of the system conditions on the statistical behavior and identify the critical regime. For this purpose, we consider four different fluctuation measures namely, density of states (DOS), reduced partition number (P/L),

the peak position of the NNSD and the dc electrical conductivity which can briefly be described as follows.

Spectral measures: A well-known measure to analyse the short range correlations among energy levels is the nearest neighbour spacing distribution (NNSD) which describes the probability $P(s)$ of two nearest neighbour energy levels to be found at a distance s measured in the units of the mean level spacing around the desired energy regime. For the weak disorder regime with extended eigenfunctions, $P(s)$ is given by the Wigner surmise

$$P_W(s) = \frac{\pi}{2}s \exp\left(-\frac{\pi}{4}s^2\right). \quad (15)$$

For the opposite limit of strong disorder with fully localized wavefunctions, $P(s)$ follows Poisson distribution

$$P_P(s) = \exp(-s). \quad (16)$$

To compare the level spacing distribution for the entire transition within an arbitrary energy regime for different lattices, we use a traditional measure, namely, the cumulative nearest neighbor spacing distribution η_i which depends on the tail behaviour of the nearest neighbour spacing distribution and is defined as

$$\eta_i = \frac{\int_{s_i}^{\infty} P(s, \Lambda) ds - \int_{s_i}^{\infty} P_W(s) ds}{\int_{s_i}^{\infty} P_P(s) ds - \int_{s_i}^{\infty} P_W(s) ds}, \quad (17)$$

where s_i , $i = 1, 2$ refer to the two crossing points of $P_W(s)$ and $P_P(s)$: $s_1 = 0.473$ and $s_2 = 2.002^{21,23,24}$. As the system makes a transition from delocalized to localized state, η_i changes from 0 to 1. While NNSD gives the short range correlations of the energy levels, there is another measure which gives the long range correlations, namely, the number variance. It is defined as the variance of the number n of unfolded energy levels in an energy interval r centered at the energy regime of interest i.e., the number variance $\Sigma^2(E, r) = \langle (n(E, r) - \langle n(E, r) \rangle)^2 \rangle$.

Participation number: The dependence of Λ on the localization length ξ through Δ_l results in sensitivity of the energy level statistics to eigenfunction behaviour. This motivates to analyse a standard measure for the eigenfunction localization, namely, the reduced participation number, referred here as P/L , which characterize the spread of eigenfunctions in the site basis. The partition number P for a wavefunction ψ_n is defined as $P^{-1} = \sum_i^N |\psi_{in}|^4$ and is proportional to the localization length.

DC conductivity: The localization of electronic wavefunction can be characterized by the DC electrical conductivity σ which can be measured also experimentally for real systems. Here we numerically calculate σ using Kubo-Greenwood formula (see Ref. 25 for details)

$$\sigma(E_F) = \frac{2\pi\hbar}{\Omega} \text{Tr} [\mathbf{J} \delta(E_F \mathbf{I} - \mathbf{H}) \mathbf{J} \delta(E_F \mathbf{I} - \mathbf{H})] \quad (18)$$

where E_F is the Fermi energy, \mathbf{H} is the Hamiltonian, Ω is the volume of the system and \mathbf{J} is the one electron

current operator

$$\bar{\mathbf{J}} = \frac{ieV}{\hbar} \sum_{\langle i,j \rangle} (\mathbf{R}_i - \mathbf{R}_j) (|i\rangle\langle j| - |j\rangle\langle i|) \quad (19)$$

with \mathbf{R}_i as the position vector of site i .

IV. NUMERICAL ANALYSIS AND RESULTS

To study the influence of system conditions on the statistical behavior, we apply exact diagonalization technique to numerically obtain the eigenvalues and the eigenfunctions of the Hamiltonian (1) for four different lattice types. For each lattice type, an ensemble of approximately 1000 realizations is considered to attain statistical accuracy. To explore the size-dependence of the statistics and its sensitivity to the energy range, many system sizes, varying from $L = 40$ to $L = 100$, are analysed for each lattice type and for two energy ranges. The latter correspond to two filling of the Hamiltonian: firstly, the half-filling which is a more natural choice of the systems like graphene, gold etc., and secondly, a filling where the density of states is significantly high, making a rigorous statistical analysis of energy levels more feasible. For the local fluctuations analysis, it is necessary to first rescale the energy eigenvalues by the local mean level spacing²⁶ (also known as *unfolding* of the levels). The average density of states for this purpose is calculated using the binning method. To analyse the local spectral fluctuations in the desired energy-range, approximately $\sim 3\%$ of the rescaled energy levels are taken around the chosen Fermi energy. The rescaled energy levels and normalized eigenfunctions are then used to calculate the density of states (DOS), reduced partition number (P/L), the peak position of the NNSD and the dc electrical conductivity. The results are arranged in order of increase in the coordination numbers in the subsections A, B, C and D containing honeycomb, kagomé, square and triangular lattices respectively. Here we describe three different methods of calculation in detail for the honeycomb lattice only; the calculations for other lattice systems are carried out following similar methods.

A. Honeycomb lattice

The 2D honeycomb structure is one of the most interesting lattice types which occurs in many systems of industrial importance e.g., graphene. In this case, the non-interacting electronic models like uniform Anderson model or tight binding model have two unique Dirac cones in a Brillouin zone (BZ). The DOS $\rho(\epsilon)$ for a 2D honeycomb lattice in the clean limit (i.e. absence of disorder) is given by

$$\rho(\epsilon) = -\frac{1}{\pi} \text{Im} \left[\lim_{\Delta\epsilon \rightarrow 0^+} \int_{1^{\text{st}} \text{BZ}} d\mathbf{k} \frac{V}{V^2 - |\mu|^2 t^2} \right], \quad (20)$$

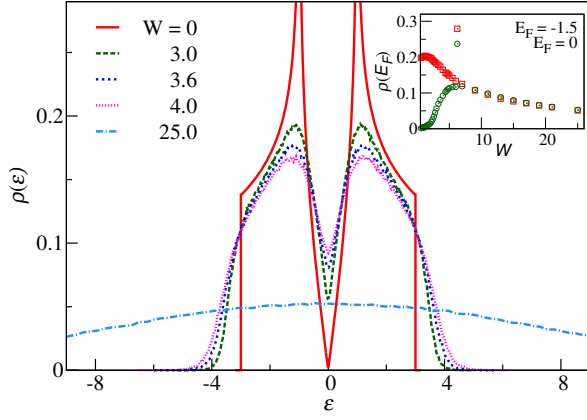


FIG. 2. (Color online) The DOS of the honeycomb lattice. The solid line shows the DOS in the absence of disorder. Three disorders strength are chosen around the critical disorder (see table I). The DOS at very large disorder is also shown. The inset shows the variation of DOS with disorder at two energy regimes.

where $V = \epsilon + i\Delta\epsilon$ and $\mu = \exp(ik_x a) + \exp[i(-k_x a/2 + k_y \sqrt{3}a/2)] + \exp[i(-k_x a/2 - k_y \sqrt{3}a/2)]$ with k_x and k_y as the components of the wave vector²⁷; $\rho(\epsilon)$ in this case vanishes at $\epsilon = \pm 3t$. In half-filled limit, the system shows a gapless state and a linear dispersion relation which can be mapped to that of a massless Dirac state. In real materials, the vacancies can lead to coupling of cones but coupling strength remains very small, the two cones being separated by a large momentum vector²⁸. The microscopic calculation for the two valley Hamiltonian indicates a crossover from anti-localization to weak localization²⁸. A recent experiment also confirms the anti-localization and weak localization of states at low and high carrier concentration respectively (for defects like charge impurity etc.).

To observe the effect of disorder, the DOS $\rho(\epsilon)$ is calculated numerically for different disorders; the results are shown in Fig. 2. In the clean system ($W = 0$), $\rho(\epsilon)$ is vanishingly small near energy $\epsilon = 0$ (the Fermi-energy at half filling) and has van-Hove singularities at $\epsilon = \pm 3t$. The effect of varying disorder on $\rho(\epsilon)$ at $\epsilon = -1.5t$ (i.e., at the bulk of DOS) and at $\epsilon = 0$ (i.e., at the half filling) is shown in the inset of Fig. 2. At the half filled energy regime, $\rho(\epsilon)$ increases till $W = 7.0t$ and then decrease exponentially as shown in the inset of the Fig. 2. In large W limit, DOS becomes flat as shown in the Fig. 2. As clear from the Fig. 2, the disorder has a significant impact on the DOS of this lattice type in the energy regime near $\epsilon = 0$ (with DOS showing a prominent dip for small disorder) and $\epsilon = \pm 1.5t$. For the fluctuations analysis we therefore, choose Fermi energy from these two regimes, namely, $\epsilon = 0$ (half filling) and $\epsilon = -1.5t$ (bulk).

Our next step is to seek the critical disorder for the extended to localized state transition in the honeycomb lattice by three different routes. While the transition in principle takes place in the thermodynamics limit $L \rightarrow \infty$

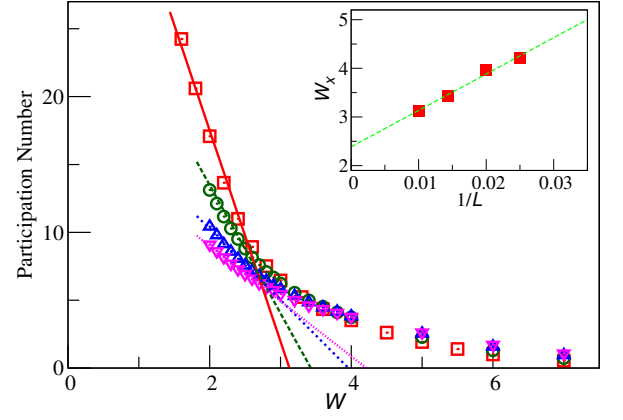


FIG. 3. (Color online) The partition number per unit length with disorder for different system sizes are shown at (a) $E_F = -1.5t$ and (b) $E_F = 0$. The linear fits of the curves cut the disorder axis at W_x . The insets of both (a) and (b) show the critical disorder W_x with $1/L$ which are fitted with the lines: $W_x = 3.13 + 68.6/L$ for inset (a) and $W_x = 2.64 + 72.6/L$ for inset (b).

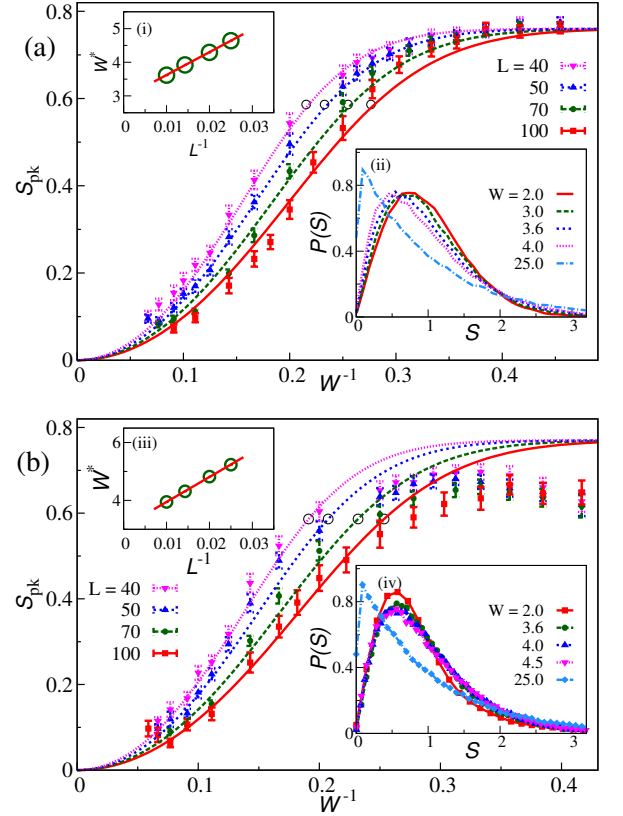


FIG. 4. (Color online) The peak-position of the NNSD with the inverse of the disorder (a) at $E_F = -1.5t$ and (b) at $E_F = 0$. The lines in the main part of (a) and (b) are the fitted curves with the function $0.77 \tanh((W^*/W)^2)$. The critical disorder for various system sizes are depicted in insets (i) and (iii). The insets (ii) and (iv) show the NNSD at different disorders for the system size $L = 100$

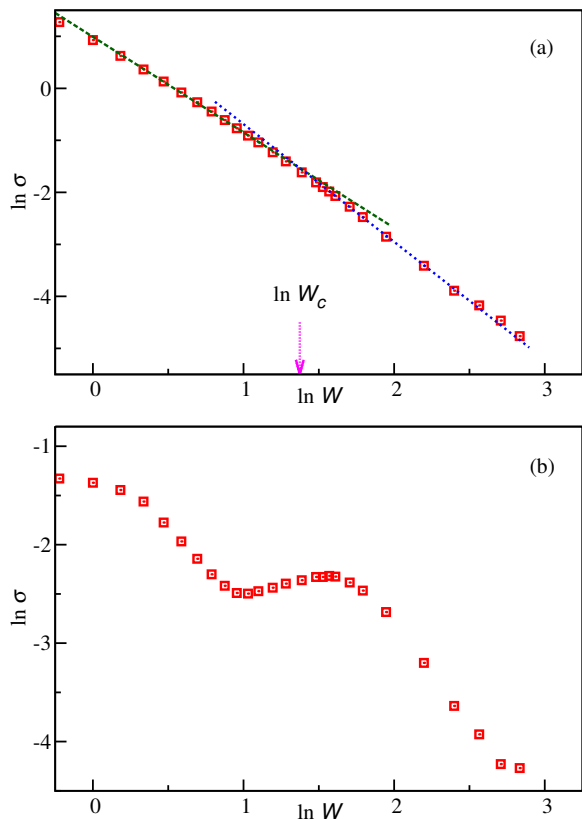


FIG. 5. (Color online) The dc conductivity with disorder at (a) $\epsilon = -1.5t$ and (b) $\epsilon = 0$. The upper part in (b) is fitted with: $\ln \sigma = 0.99 - 1.84 \ln W$ and the lower part is fitted with $\ln \sigma = 1.577 - 2.265 \ln W$ which gives $W_c = 3.9t$.

at a critical disorder (or critical energy), the finite systems undergo a smooth crossover within a critical regime (around critical disorder or energy). It is therefore imperative to analyze the critical disorder $W_{crit}(L)$ for many finite system sizes, with exact critical disorder given by $\lim_{L \rightarrow \infty} W_{crit}(L) = W_{crit}$. For this purpose, we first analyse the disorder dependence of the reduced participation number P/L . Fig. 3 elucidates the behavior of the reduced partition number at two Fermi energies at (a) the bulk and (b) at the half filling respectively. For each case P/L varies linearly with the disorder W in the small disorder regime while it varies exponentially for strong disorders. The linear portion of the curves are fitted with lines ($P/L = a - bW$) which cut the disorder axis at $W_x = a/b$. For $W > W_x$, the localization length of the systems is less than the system size. At $W = W_x$, the localization length become equal to the system size L , thus suggesting W_x as the finite size critical disorder (more clearly $W_{crit}^{pn}(L) = W_x$ with superscript referring to the method applied). (Note, for later reference, here we use different notations for the critical disorder obtained by different methods). The critical disorder measured from this method for different system sizes of the honeycomb lattice are given in the table I. As shown in the insets

of Fig. 3(a) and (b), W_x for different system sizes vary linearly with the inverse of the system size. The numerics reveals the sensitivity of W_x to Fermi energy too: for $L = 100$, the value of W_x is 3.4 at the half filling and 3.8 at the bulk of the DOS.

Continuing with our quest for critical disorder, our next step is to analyze the Nearest neighbor level spacing distribution (NNSD) defined in section III. The decrease of disorder in the finite systems causes the NNSD to crossover from the Poisson distribution (eq.(16)) to the Wigner surmise (eq.(15)). To analyse the NNSD for different disorder strengths, the energy eigenvalues are apriori unfolded using the local mean level spacing. The numerically obtained NNSD for a given disorder and system size is compared with the Brody distribution²⁹:

$$P_B(S) = \alpha(1 + \omega)S^\omega \exp(-bS^{1+\omega}) \quad (21)$$

which gives the Brody parameters ω and b as a function of W and L . The fitted Brody distribution is then used to calculate the peak position dependence on the disorder strengths W for a fixed size L . The dependence is plotted with respect to inverse of the disorder in Fig. 4 for many system sizes and at two Fermi energies (the bulk and the half filling). The curve depicting peak positions with respect to disorder for each L is now fitted with the function

$$F(W) = 0.77 \tanh((W^*/W)^n) \quad (22)$$

where $n = 2$ and W^* is the critical disorder strength (in this method) at which localization occurs for a finite system size L (more clearly $W_{crit}^{nnsd}(L) = W^*$). This type of functional dependence of the peak position of NNSD was suggested by A J Millis *et al.* in context of the energy level statistics of one dimensional spin-1/2 chain to find the critical value J_c indicating integrable and non-integrable boundary³⁰. As shown in the main Fig. 4 (b), there is large deviation in the peak position-disorder curve from the function $F(w)$ in the small disorder regime for the half filled case. $F(W)$ with $n = 2$ however fits well in small W limit for the case where the Fermi energy is chosen away from the half filling. As shown in the insets (i) and (iii) of Fig. 4 (a) and (b) respectively, the crossover disorder W^* decreases with system size L . Again W^* shows an energy dependence for a fixed L : for $L = 100$, $W^* = 3.9$ and 3.6 for $\epsilon = 0$ and $\epsilon = -1.5t$ respectively (see table I). The insets (ii) and (iv) of Fig. 4 (a) and (b) also display the NNSD behavior for $\epsilon = 0$ and $\epsilon = -1.5t$ respectively. We notice that for $\epsilon = 0$ or half filled case the NNSD vary significantly in small impurity limit but the variation in the bulk limit $\epsilon = -1.5t$ is weak below $W = W^*$. The unusual behaviour can be explained from the rapid change in the DOS with disorder at the half filling. The half filling case suggests that the system is still in the ballistic limit for small disorder.

The electronic conductivity is an important characteristic of the localization to delocalization transition, with high conductivity an indicator of the large localization

ϵ	L	W_x	W^*	W_c
$-1.5t$	40	4.9	5.2	
	50	4.4	4.8	
	70	4.2	4.3	
	100	3.8	3.9	3.9

TABLE I. Critical disorders calculated from all three methods for the honeycomb lattice at two different Fermi energies for four system sizes

length. The dc electrical conductivity for various system sizes for the two energy ranges are calculated using the Kubo-Greenwood formula. The minimum conductivity σ_m of a clean sample at $T = 0$ and half-filled limit is analytically predicted to be $4e^2/\pi h$; this is also confirmed by our analysis. The effect of changing disorder on the behavior of conductivity (σ) in honeycomb lattice is displayed in the figure 5 (a) and (b) at the bulk of DOS and at the half filling respectively. In the bulk limit, the conductivity decreases with increasing disorder following a power law dependence with exponents -1.84 in the weak disorder regime and -2.265 in the strong disorder regime. The intersection of these two lines gives the crossover disorder $W_{crit}^{cond.}(L) = W_c = 3.9t$ which is almost in agreement with the values calculated from the other two methods. The disorder-dependence of conductivity at the half filling however shows some atypical behavior: it remains almost constant at very small disorders up to $W = 0.4t$, decreases thereafter up to $W < 2.8t$, then increases till $W = 5.0$ and decrease afterwards for further increase in W . A possible explanation of this flip-flop behavior may lie in poor statistics due to small number of states available at this filling or due to dominant finite size effect. We intend to probe this behavior with more numerical rigor separately.

Next we compare the crossover values of $W_{crit}(L)$ calculated from three different methods (i.e W_x, W^*, W_c) for four different L . Due to statistical reliability, the comparison is shown in table I (only for the bulk energy limit). As given in the table I, W_x and W^* decrease with an increase in the system size L , varying as L^{-1} (As the size dependence on the conductivity is negligibly small for this case, we present the data for $L = 100$ case only). This clearly indicates conductivity as a better measure to find the critical disorder for the transition (due to faster convergence with L). The analysis suggests the critical disorder $W_{crit} \approx 3.8$.

B. Kagomé lattice

The DOS $\rho(\epsilon)$ of the Anderson model with uniform site energy for a kagomé lattice is exactly solvable in the clean limit, and is composed of the two bands of the honeycomb lattice shifted in energy by t with amplitude reduced by a factor of $2/3$ and a delta peak at $\epsilon = 2t$ with

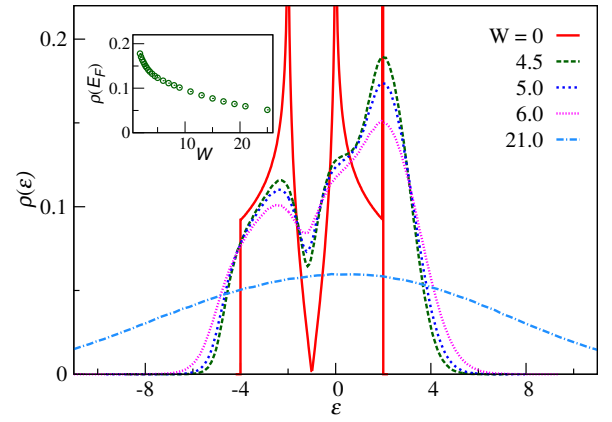


FIG. 6. (Color online) The DOS of the kagomé lattice for different disorders. The solid line is the DOS for the clean system. The inset shows the DOS at half filling with the disorder.

weight $1/3$. The DOS diverges at three values of energy (at $\epsilon = 0$ and $\pm 2t$) and is zero for $\epsilon = -t$ for the clean system³¹. The band width in the clean limit of this lattice is spanned from $\epsilon = -4t$ to $\epsilon = 2t$. Effect of the onsite disorder on the DOS of the kagomé lattice Hamiltonian of system size L^2 with $L \sim 100$ is shown in the Fig. 6 for three disorder strengths alongwith the clean limit. As clear from the figure, the singularities in the DOS vanish with the increase in disorder, with DOS approaching a Gaussian distribution in energy at very large disorder. The inset in Fig 6 displays the behavior of the DOS at the Fermi energy for varying disorders which turns out to be an exponential decay in large disorder limit.

To determine the critical disorder W_{crit} in this case, we again apply the three methods mentioned in detail for the honeycomb lattice. The results for reduced par-

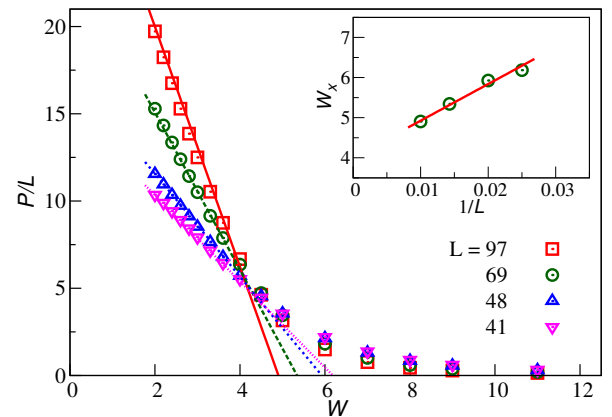


FIG. 7. (Color online) The reduced partition number for different system sizes of the kagomé lattice. The linear portion of the curves are fitted with lines which cut the disorder axis at W_x . The inset shows the critical disorder W_x with the inverse of the system size and which follows the fitted line: $W_x = 4.0 + 91.3/L$.

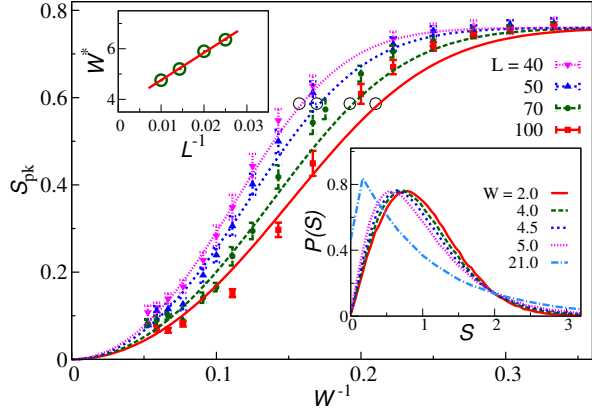


FIG. 8. (Color online) The peak-position of the NNSD vs the inverse of the disorder. The lines in the main figure are fitted curves with $0.77 \tanh((W^*/W)^2)$. The top left inset shows the critical disorder with the system size which follows: $W^* = 3.67 + 108.5/L$. The bottom right inset shows the NNSD of the kagomé lattice of system size $L \sim 100$ for different disorders

icipation number P/L are displayed in Fig. 7 for different system sizes in the kagomé lattice. As shown in the inset of Fig. 7, the finite size critical disorder W_x calculated from this method varies linearly with the inverse of the system size. The results for the search of critical disorder through the NNSD peak position analysis are displayed in Fig. 8, for four different system sizes of kagomé lattice. Here again the curve describing peak position-disorder dependence is fitted with the function $F(W)$ given by Eq.(22) with $n = 2$. As shown in the inset of Fig. 8, W^* decrease with system size L . The behavior of NNSD shown in the inset of Fig. 8 indicates a very small variance in NNSD for $W < W^*$.

The disorder dependence of the electronic conductivity for kagomé lattice is displayed in Fig. 9 (shown here only for $L = 100$); it reveals a decreasing conductivity with

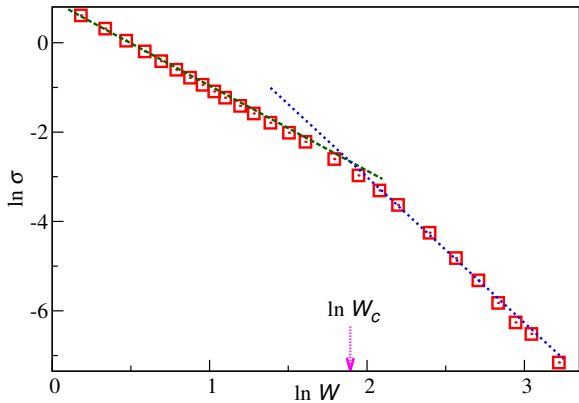


FIG. 9. (Color online) DC conductivity of the kagomé lattice of linear size ≈ 100 with disorder. The upper part of the plot follows: $\ln \sigma = 0.938 - 1.9 \ln W$ and the lower part follows: $\ln \sigma = 3.514 - 3.2 \ln W$ and the critical disorder $W_c = 5.9$.

increase of disorder, along with two different power law behaviors in small and large disorder ranges (indicated by two linear regimes with different slopes in the $\log \sigma$ - $\log W$ plot). For a system size $L \approx 100$, the crossing point of the two lines lies at $W_c \sim 5.9t$.

Table II displays the critical disorder $W_{crit}(L)$ calculated from three different methods for four different system sizes. Similar to the honeycomb lattice case; the size dependence of W_c is not presented as it is negligibly small for this system also. whereas W_x and W^* calculated from participation number and shift in peak height of NNSD is linear with L^{-1} . Similar to the case of honeycomb lattice, here again the calculations suggest a convergence of W_c to its critical value lattice, once again confirming a weaker sensitivity of the conductivity approach to finiteness of the kagomé lattice.

L	W_x	W^*	W_c
40	6.2	6.4	
50	5.9	5.9	5.9
70	5.3	5.2	
100	4.9	4.8	

TABLE II. Critical disorders calculated from all three methods for the kagomé lattice for four system sizes

C. Square lattice

The DOS $\rho(\epsilon)$ of an Anderson model with uniform site energy for a square lattice at the clean limit can be calculated from the band dispersion relation

$$\epsilon(\mathbf{k}) = 2t(\cos k_x + \cos k_y). \quad (23)$$

In the clean limit, the DOS has a van Hove singularity at

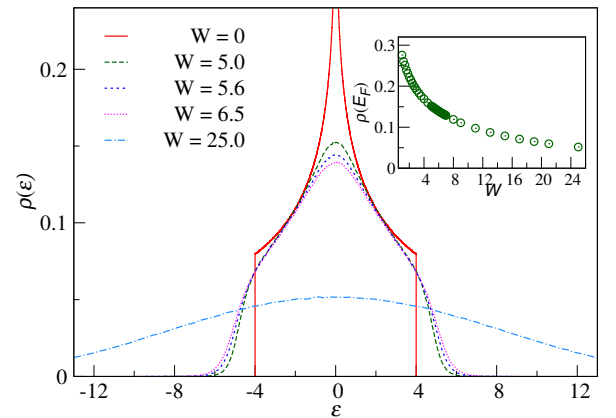


FIG. 10. (Color online) The DOS of the square lattice at different disorders. The solid line is the DOS for a clean system. The inset shows the DOS at the half filling with the disorder.

$\epsilon = 0$ and vanishes for $|\epsilon| > 4t$. Effect of on-site disorder on the DOS for four system sizes of square lattices is shown in the Fig. 10. As shown in the inset of Fig 10 the DOS at half filling decays exponentially with disorder which is consistent with the Ref. 32.

Following the same procedure as mentioned in the case of honeycomb lattice, here again we analyze the disorder dependence of the reduced partition number P/L , NNSD and DC conductivity for different system sizes but only in bulk energy regime; the results are shown in Fig. 11, Fig. 12 and Fig. 13 respectively. As clear from the figures, the qualitative behavior in this case is same as in the bulk energy regimes of honeycomb and kagome lattices; a quantitative difference however shows up in the fitted line $W_x = 5.39 + 106.8/L$ for P/L , with power n for fit $F(W)$ as 3 (instead of 2 as in previous two cases), with $W^* = 5.38 + 118.4/L$ and $W_c = 6.5t$. The critical disorders W_x, W^* and W_c from the three methods are given in table III. W_x, W^* and W_c for this case show an inverse linear dependence on the system size.

L	W_x	W^*	W_c
40	8.0	8.3	
50	7.6	7.8	7.6
70	6.9	7.0	7.0
100	6.5	6.6	6.5

TABLE III. Critical disorders calculated from all three methods for the square lattice at half filling for four system sizes

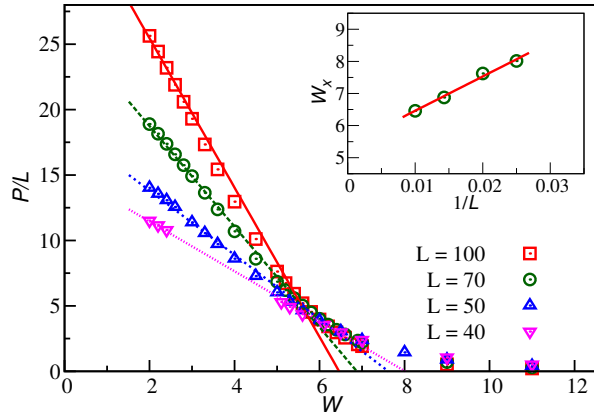


FIG. 11. (Color online) The reduced partition number of the square lattice of different system sizes. The upper part of the curves are fitted with lines which cut the disorder axis at W_x . The inset shows the critical disorder W_x with the inverse of the system size and the fitted line: $W_x = 5.39 + 106.8/L$.

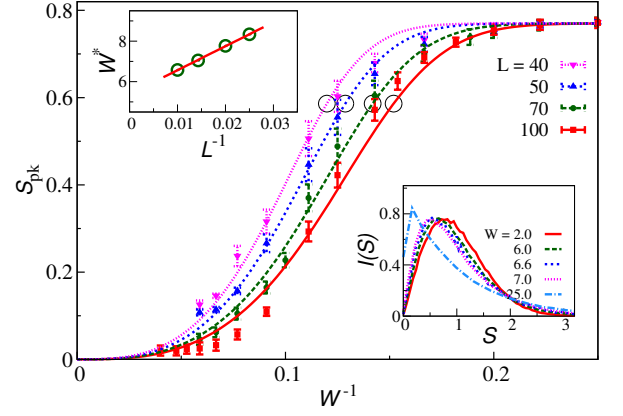


FIG. 12. (Color online) The peak position of the NNSD of the square lattices of various system sizes. The curves in the main plot are fitted with: $0.77 \tanh((W^*/W)^3)$. The top left inset W^* are plotted with inverse of the system size which follows the line: $W^* = 5.38 + 118.4/L$. The bottom right inset depicts the NNSD at different disorder for the square lattice of size $L = 100$.

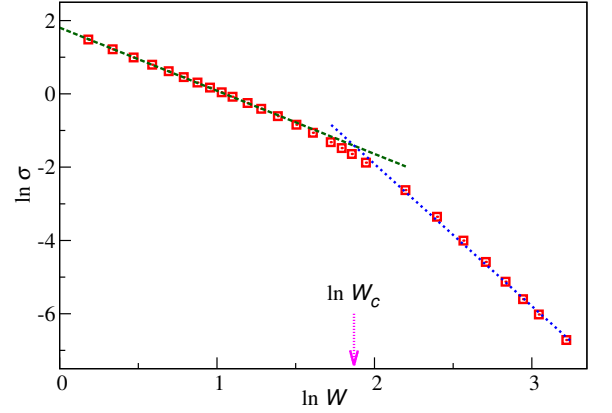


FIG. 13. (Color online) DC conductivity of the disordered square lattice at the half filling. The upper part is fitted with: $\ln \sigma = 1.807 - 1.723 \ln W$ and the lower part is fitted with: $\ln \sigma = 5.845 - 3.879 \ln W$ which gives $W_c = 6.5t$.

D. Triangular lattice

In the clean limit, the DOS $\rho(\epsilon)$ for a triangular lattice can exactly be obtained from the band dispersion relation

$$\epsilon(\mathbf{k}) = -t \left[2 \cos(k_x) + 4 \cos\left(\frac{k_x}{2}\right) \cos\left(\frac{\sqrt{3}}{2}k_y\right) \right], \quad (24)$$

where \mathbf{k} is confined to the first Brillouin Zone³¹. The DOS obtained numerically in the clean limit along with the presence of disorder is shown in the Fig 14. The analysis indicates a van Hove singularity in the DOS at energy $E = 2t$ and it approaches zero for $\epsilon < -6t$ and $\epsilon > 3t$. The energy dependence of the DOS for four disorders is shown in the Fig. 14. In the inset of Fig 14, the disorder dependence of the DOS at a fixed energy (half

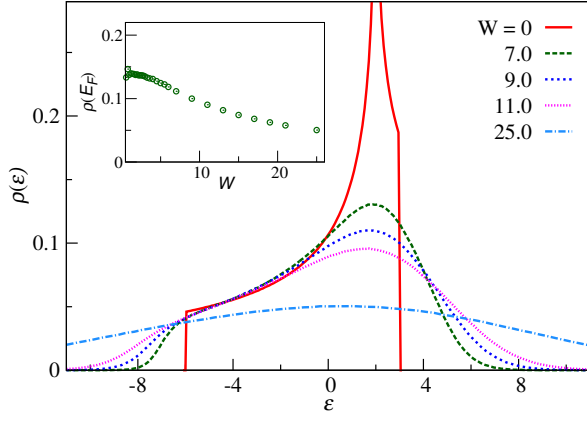


FIG. 14. (Color online) The DOS of the triangular lattice in presence of different disorder strengths. The solid line is the DOS for the system without any disorder. The inset shows the variation of DOS at the half filling with disorder.

filled energy) is displayed which indicates an exponential decrease in DOS with increasing disorder.

The results for the search of critical disorder in triangular lattice by the three methods (same as mentioned above for previous lattices) are shown in Fig. 15, Fig. 16 and Fig. 17 respectively. The figures again indicate the same qualitative behavior as in the bulk energy regimes of honeycomb, kagomé and square lattices but the quantitative difference shows up in the fitted line $W_x = 6.88 + 169.3/L$ for P/L , with power n for fit $F(W)$ as 3 (same as in square lattice but different from honeycomb and kagomé), with $W^* = 7.10 + 165.8/L$ and $W_c = 8.7t$. A comparison of critical disorders W_x , W^* and W_c (for Fermi energy in bulk) from the three methods is displayed in table IV which confirms, as for previous three lattice types, an inverse linear dependence for

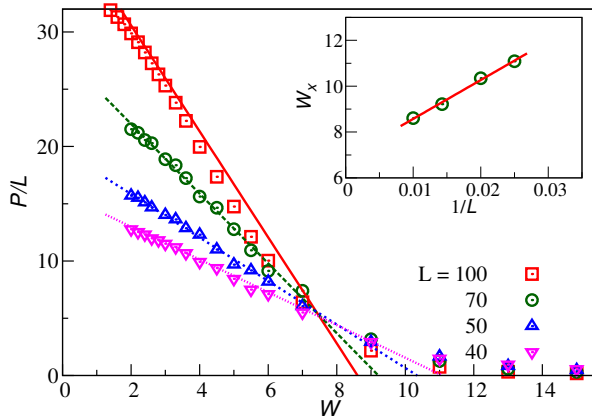


FIG. 15. (Color online) The reduced partition number of triangular lattice of different system sizes with disorder. The are the linear fit for the reduced partition number curve which cut the W axis at W_x . The finite size dependence of the critical disorder W_x is shown in the inset which follows the line: $W_x = 6.88 + 169.3/L$.

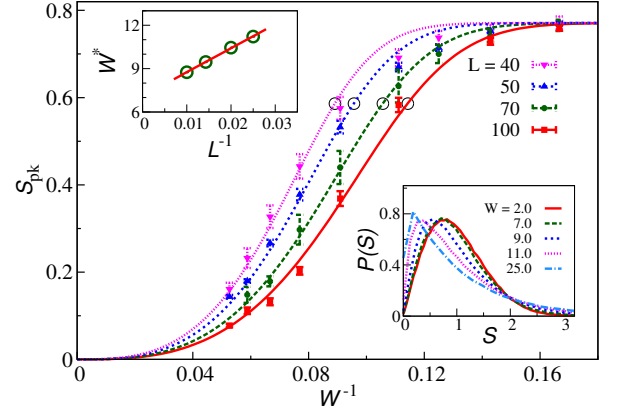


FIG. 16. (Color online) The peak position of the NNSD with the inverse disorder for the triangular lattice of different system sizes. The data is fitted with the function $0.77 \tanh((W^*/W)^2)$. The top left inset shows the variation of W^* with system size which follows the fitted line: $W^* = 7.10 + 165.8/L$. The bottom right inset shows the NNSD of the triangular lattice of system size $L = 100$ calculated at the half filling.

W_x , W^* and W_c in this case too.

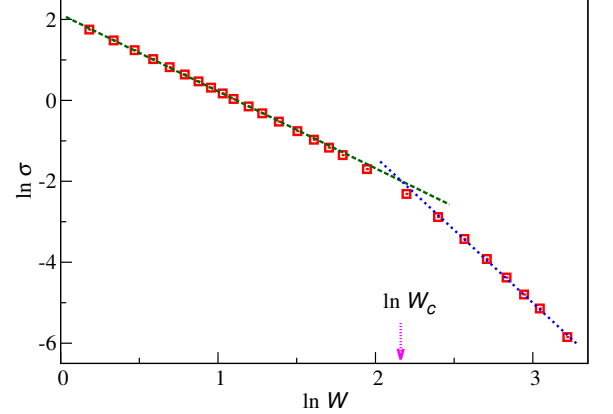


FIG. 17. (Color online) DC conductivity of the disordered triangular lattice at the half filling. The upper part is fitted with: $\ln \sigma = 2.126 - 1.902 \ln W$ and the lower part is fitted with: $\ln \sigma = 5.812 - 3.606 \ln W$. The crossing point of these two fitted curves gives $W_c = 8.7$.

L	W_x	W^*	W_c
40	11.1	11.2	
50	10.3	10.4	10.4
70	9.2	9.5	9.4
100	8.6	8.7	8.7

TABLE IV. Critical disorders calculated from all three methods for the triangular lattice at half filling for four system sizes

V. COMPLEXITY PARAMETER FORMULATION OF THE TRANSITION

As discussed in previous section, the numerical analysis reveals the qualitative insensitivity of the local fluctuations in physical properties to system parameters (within a fixed energy range) although a quantitative dependence is indicated. More clearly, for each lattice type with Fermi energy in the bulk we observe the following behavior: (i) the reduced participation ratio has a linear/exponential dependence on the disorder in weak/strong disorder regime, respectively, (ii) the disorder dependence of the peak positions of NNSD can be described by the function $F(w)$ (with different n values), (iii) the conductivity has a power law dependence on the disorder with different exponents in weak and strong disorder regime. The observed behavior therefore strongly suggests the possibility of a common mathematical formulation of the statistical properties where system dependence enters through a single function of all system parameters. The theoretical steps for such a formulation are briefly reviewed in section II, with the function referred as the complexity parameter. As defined in Eq. 11,

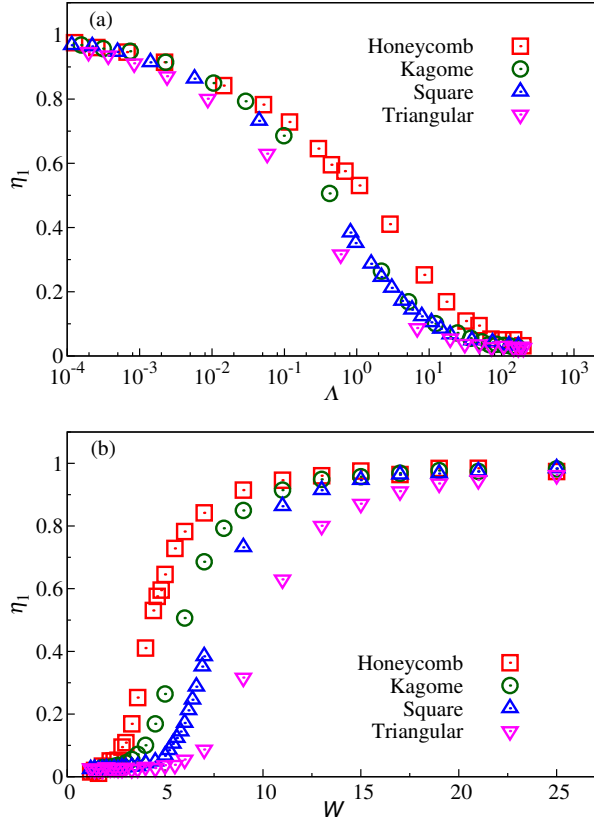


FIG. 18. (Color online) The variation of η_1 with (a) the rescaled complexity parameter Λ for four different two dimensional lattices and (b) the disorder W . As obvious, the behaviour for different lattices coincide in terms of Λ but not in terms of disorder.

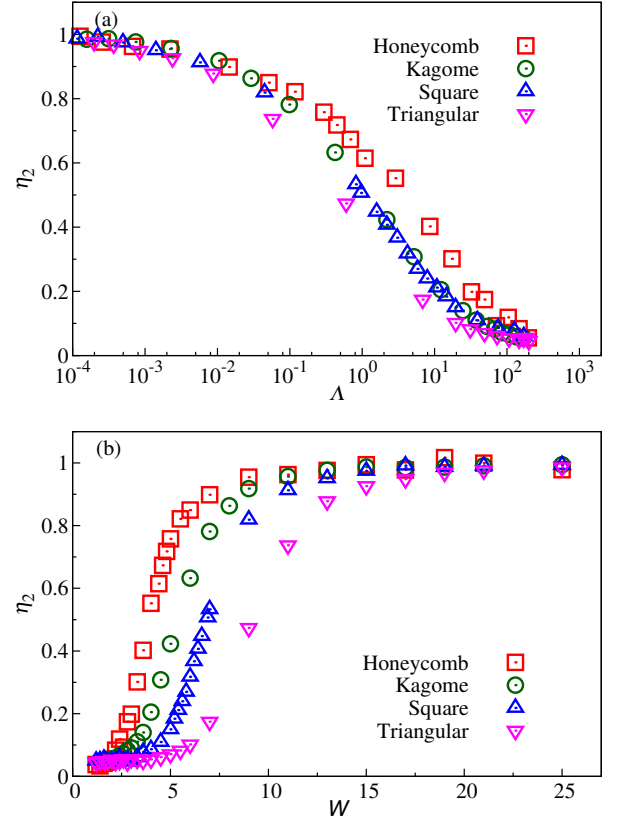


FIG. 19. (Color online) The variation of η_2 with (a) the rescaled complexity parameter Λ for four different two dimensional lattices and (b) the disorder W . As obvious, the behaviour for different lattices coincide in terms of Λ but not in terms of disorder.

the function Y is a combination of various system parameters, with explicit dependence on the disorder strength, hopping and system size. The information about dimensionality and boundary conditions is implicitly contained in the sparsity of the matrix H as well matrix element identities and therefore in the summation in Eq. 10. The necessary rescaling of energy levels for comparison of the fluctuations however leads to Λ (Eq. 14) as the relevant transition parameter; the rescaling therefore introduces the crucial dependence on the dimensionality as well as on the Fermi energy. The obvious relevance of the theoretically obtained single parameter governing the transition renders its numerical/ experimental verification very desirable. For this purpose, we consider here three well-known spectral fluctuation measures namely, the cumulative NNSD η_1, η_2 and the number variance $\Sigma^2(r)$ (defined in section III) of the four lattices and analyze their evolution in terms of the complexity parameter Λ instead of disorder W . The verification in context of the eigenfunction fluctuations is yet to be carried out and will be reported elsewhere.

To calculate Λ for our analysis, we use the fact that the localization length ξ is proportional to the average partic-

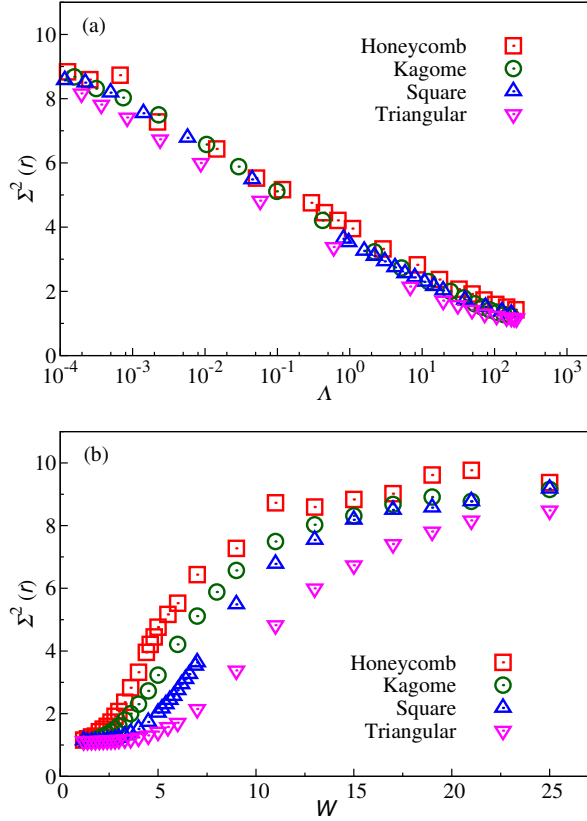


FIG. 20. (Color online) The number variance $\Sigma^2(r)$ at $r = 10$ with (a) the rescaled complexity parameter Λ for four different two dimensional lattices and (b) the disorder W . As obvious, the behaviour for different lattices coincide in terms of Λ but not in terms of disorder.

ipation number P . Fig. 18, 19 and 20 show the results for η_1 , η_2 and $\Sigma^2(r)$, respectively, for four lattices for Fermi energy at $e = 0$. As clear from the part (a) of these figures, Λ -governed evolution of each of these measure for all four lattices falls almost on the same curve for entire crossover from the localization to delocalization; the difference of connectivity of the lattices does not affect their behavior. Note a disorder (W)-dependent evolution of η_1 and η_2 for honeycomb and kagome lattices is expected to differ from that of square and triangular ones (as suggested by the observed difference in disorder governed evolution of the NNSD-peaks of the lattices, with η_1 and η_2 being cumulative NNSDs); the deviation of the results for four lattices is clearly visible from part (b) of Fig. 18, 19 and 20). This clearly reveals Λ as the parameter in terms of which the transition in spectral statistics in Anderson lattices follows a universal route.

VI. DISCUSSION

From our results it is clear that the statistical behavior of two dimensional finite size lattices depends on the

coordination number, the lattice connectivity and the system size. The statistics is different for the lattices with same coordination number with different connectivity which is evident from the spectral statistics of the square lattice and the kagomé lattice (both having same coordination number). The spectral averaged density of states for all the lattices, considered in this paper, show strong disorder sensitivity in weak disorder regime, and, at least one van-Hove singularity in clean limit. The position of the singularity is sensitive to the lattice type; it occurs at energy $\epsilon = 0$ for the square lattice, at $\epsilon = 2.0t$ for the triangular lattice, at $\epsilon = \pm t$ for the honeycomb lattice and at $\epsilon = 0, -2t$ for the kagomé lattice. The density of states for the honeycomb lattice and the kagomé lattice are related to that of the triangular lattice³¹ and can be written in terms of the DOS of the triangular lattice in the clean limit. The DOS at $\epsilon = \epsilon_F$ varies differently for different lattice systems at small disorder region whereas they all decay exponentially for strong disorder region. At very large disorder, the DOS for all the lattice system lead to a Gaussian distribution.

Three methods are used to estimate the critical disorder for delocalization to localization transition in two dimensional finite size lattices. First, the reduced partition number (P/L) are used to find the critical disorder W_x . P/L varies linearly in the weak disorder regime whereas, it varies exponential in the strong disorder regime. Next, we study the peak position of the NNSD of each lattice type for four system sizes; which fits with function $0.77 \tanh((W^*/W)^\alpha)$ where $\alpha = 3$ for the square and the triangular lattices whereas, $\alpha = 2$ for the honeycomb and the kagomé lattices. Last, the Kubo-Greenwood dc conductivity is to find the critical disorder (W_c) as the conductivity follows two power law decay in the weak and strong disorder regimes. The critical disorders calculated from all the methods are in agreement with each other.

We also analyze dependence of the critical disorder on finite size and lattice structure considering four system sizes for all the cases. Our results indicate (i) a linear variation of critical disorder with $1/L$, (ii) it is smallest for the honeycomb lattice and largest for the triangular lattice and increases with the coordination number for a finite lattice of size L . Although the coordination number is same for the kagomé lattice and the square lattice the critical disorders are different for the two cases. Therefore, the critical disorder depends not only on the coordination number but also on the lattice connectivity.

Finally, we compare three fluctuation measures namely, the cumulative NNSD measures η_1 , η_2 and the number variance $\Sigma^2(r)$ for the four lattices and study their evolution with the single complexity parameter Λ . The results confirm the single parameter dependence of the localization to delocalization transition in Anderson Hamiltonian in context of the spectral statistics.

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